Channel Estimation Techniques

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1 LTI Channel Model

For simplicity, let us assume that the channel is linear and time-invariant. Later, we'll extend the model to time-invariant channel.

Let the input sequence of information bits be $s[n]$. Let $v[n]$ be additive noise. The output $y[n]$ at the receiver can be modelled as

$$y[n] = \sum_{k=0}^{L-1} h[k]s[n-k] + v[n] \qquad (1)$$

where $n = 0, 1, \ldots, N-1$. We have assumed that the channel impulse response is $h[k]$.

The *channel estimator* is defined as any static (known function of observable random variables that is itself a random variable) whose values are used to estimate $\tau(\theta)$, where $\tau(\cdot)$ is some funtion of an unknown parameter θ.

2 Maximum Likelihood Estimator

The channel vector is given by

$$\mathbf{h} = \{h[0], h[1], \cdots, h[L-1]\}^T \tag{2}$$

where $\{\cdot\}^T$ is the transpose of a vector. Suppose that we received N samples of the output

$$\mathbf{y} = \{y[0], y[1], \cdots, y[N-1]\}^T \tag{3}$$

we can rewrite the Eq(1) as

$$\mathbf{y} = \mathbf{S}\mathbf{h} + \mathbf{v} \tag{4}$$

The matrix $\mathbf{S}$ is an $N \times L$ Toeplitz matrix consisting of the samples of the input sequence $\{s[n], n = 0, 1, \ldots, N-1\}$ given by

$$\mathbf{S} = \begin{bmatrix} s[0] & s[N-1] & \cdots & s[N-L+1] \\ s[1] & s[0] & \cdots & s[N-L+2] \\ \vdots & \vdots & \ddots & \vdots \\ s[N-1] & s[N-2] & \cdots & s[0] \end{bmatrix} \tag{5}$$

Let θ be the vector of unknown parameters that may contain the channel vector $\mathbf{h}$ and possibly the entire or part of the input vector $\mathbf{s}$. Assume that the joint probability distribution of noise vector $\mathbf{v}$ and the input vector $\mathbf{s}$ is known. We can then obtain the probability density function (pdf) of the observation vector $\mathbf{y}$.

The joint pdf of the observation $f_{\mathbf{y}}(\mathbf{y}; \theta)$ is called as the *likelihood function*. For simplicity, assume that θ is a single parameter. If the observation samples $y[0], y[1], \ldots, y[N-1]$ are independent of each other, the likelihood function can be written as the product of individual densities:

$$f_{\mathbf{y}}(\mathbf{y}; \theta) = f(y[0], \theta) f(y[1], \theta) \cdots f(y[N-1], \theta) \tag{6}$$

Thus, the maximum likelihood estimator is the solution of the equation

$$\frac{df_{\mathbf{y}}(\mathbf{y}; \theta)}{d\theta} = 0 \tag{7}$$

Generally, it is easier to find the maximum of the logarithm of the likelihood, $\ln f_{\mathbf{y}}(\mathbf{y}; \theta)$ because both $f_{\mathbf{y}}(\mathbf{y}; \theta)$ and $\ln f_{\mathbf{y}}(\mathbf{y}; \theta)$ have their maximums at the same value of θ. If, instead of a single unknown θ, there are multiple unknowns, the normal derivative $d/d\theta$ in Eq(7) is replaced by the partial derivative $\partial/\partial\theta$.

2.1 Example

Assume that the channel impulse response $\mathbf{h} = h[0], h[1], \ldots, h[N-1]$ is a Gaussian random process, that is

$$f_{h[n]}(h[n]; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(h[n]-\mu)^2\right] \tag{8}$$

with mean $-\infty < \mu < \infty$ and variance $\sigma^2, \sigma > 0$. Find the MLE $\hat{\mu}$ and $\hat{\sigma^2}$ of μ and σ^2 respectively.

It is reasonable to assume that the individual samples $h[0], h[1], \ldots, h[N-1]$ are independent of each other. In such a case, the likelihood function is given by

$$\begin{aligned} LF(h[n]; \mu, \sigma) &= \prod_{n=0}^{N-1} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(h[n]-\mu)^2\right] \\ &= \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(h[n]-\mu)^2\right] \end{aligned} \tag{9}$$

The logarithm of the likelihood function is given by

$$\begin{aligned} LLF &= \ln LF(h[n]; \mu, \sigma) \\ &= -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(h[n]-\mu)^2 \end{aligned} \tag{10}$$

At the maximum (or minimum) of LLF, the first derivative of LLF is 0. However, at the maximum, the second derivative of LLF is negative. In other words, we need to solve for $\mu = \hat{\mu}$ such that

$$\begin{aligned} \frac{\partial}{\partial\mu} LLF &= 0 \\ \frac{\partial^2}{\partial\mu^2} LLF &< 0 \end{aligned} \tag{11}$$

Similarly, for $\sigma^2 = \hat{\sigma}^2$

$$\begin{aligned} \frac{\partial}{\partial\phi} LLF &= 0 \\ \frac{\partial^2}{\partial\phi^2} LLF &< 0 \end{aligned} \tag{12}$$

where $\phi = \sigma^2$.

Wait until next week for the solution of above equations ! See you !

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